Calabi-Yau 3-folds from fiber products of rational quasi-elliptic surfaces

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CALABI-YAU 3-FOLDS FROM FIBER PRODUCTS OF RATIONAL QUASI-ELLIPTIC SURFACES

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1. Introduction

Let X be a smooth projective variety over an algebraically closed field k. We say X is a Calabi-Yau manifold if it satisfies $K_X \sim 0$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. We construct some Calabi-Yau 3-folds over k of characteristic $p > 0$, which have properties peculiar to positive characteristic. Here the "peculiar properties" we intend are:

(1) Supersingularity. We define X is supersingular if the height of the Artin-Mazur formal group $\Phi^3(X, \mathbb{G}_m)$ is ∞, i.e. $\Phi^3(X, \mathbb{G}_m) \cong \hat{\mathbb{G}}_a$. Supersingularity is one of the most interesting properties in algebraic varieties in positive characteristic.

(2) Non-liftablity. We say that X is liftable to characteristic zero if there exists a smooth projective morphism $\mathscr{X} \to \operatorname{Spec} R$, where R is a discrete valuation ring, such that the closed fiber is isomorphic to X, and the quotient field of R is of characteristic zero. It is known that Deligne proved that all K3 surfaces are liftable to characteristic zero. But in dimension 3, there exist some non-liftable Calabi-Yau 3-folds which were constructed by Hirokado [Hir99] and Schröer [Sch04].

(3) Having the non-smooth fibrational structure. Suppose X has a fibrational morphism $f : X \to S$ from X to a smooth projective variety S. In positive characteristic, general fiber of f might be singular while the total space X is smooth.

Our main theorems are as follows:

Theorem 1.1 (Characteristic 3). *In characteristic* 3, *we have a non-singular Calabi-Yau 3-fold X with the following properties:*

(1) *X is unirational, therefore supersingular.*

(2) $(b_0, \dots, b_6) = (1,0,20,6,20,0,1), (1,0,25,4,25,0,1),$
$(1,0,30,2,30,0,1), (1,0,35,0,35,0,1),$
$(1,0,41,0,41,0,1).$

In particular, X does not lift to characteristic 0 *if* $b_3(X) = 0$.

(3) *X admits fibrations whose general fibers are*
- *non-normal rational surface, and*
- *supersingular K3 surface with an RDP of type* A_2.

Moreover, some examples we construct have another fibration whose general fiber is

- *smooth supersingular K3 surface.*

Theorem 1.2 (Characteristic 2). *In characteristic* 2, *we have a nonsingular Calabi-Yau threefold X with the following properties:*

(1) *X is unirational, therefore supersingular.*
(2) $(b_0, \ldots, b_6) = (1,0,25,4,25,0,1), (1,0,36,2,36,0,1),$
$(1,0,47,0,47,0,1), (1,0,52,2,52,0,1),$
$(1,0,63,0,63,0,1).$
In particular, X does not lift to characteristic 0 *if* $b_3(X) = 0$.
(3) *X admits fibrations whose genenal fibers are*
- *non-normal rational surface, and*
- *smooth supersingular K3 surface.*

Moreover, some examples we construct have another fibration whose general fiber is
- *supersingular K3 surface with three RDPs of type* A_1.

For the details, see [HIS06] and [HIS].

Our method is based on the work by Schoen ([Sch88]). He constructed Calabi-Yau 3-folds as the fiber product of two rational elliptic surfaces $\varphi_i : Y_i \to \mathbb{P}^1$ $(i = 1, 2)$ with sections:

$$\begin{tikzcd} & Y_1 \times_{\mathbb{P}^1} Y_2 \arrow[dl] \arrow[dr] & \\ Y_1 \arrow[dr, "\varphi_1"'] & & Y_2 \arrow[dl, "\varphi_2"] \\ & \mathbb{P}^1. & \end{tikzcd}$$

One can see $Y_1 \times_{\mathbb{P}^1} Y_2 \in |-K_{Y_1 \times_k Y_2}|$, and we obtain a Calabi-Yau 3-fold by a small resolution $\pi : X \to Y_1 \times_{\mathbb{P}^1} Y_2$ under some conditions about singular fibers of φ_1 and φ_2.

Now we consider the case when both Y_1 and Y_2 are quasi-elliptic surfaces. A *quasi-elliptic surface* $\varphi : Y \to C$ is a nonsingular projective surface Y with a morphism to a nonsingular curve C, such that $\varphi_* \mathcal{O}_Y = \mathcal{O}_C$ and a general fiber is a rational curve with an ordinary cusp. Quasi-elliptic surfaces exist only in characteristic 2 and 3, enjoying properties analogous to elliptic surfaces. Let Σ be the closure of the nonsmooth locus of Y_η/η inside Y. We call it the moving cusp of $\varphi : Y \to C$.

We try to find a crepant resolution of singularities $\pi : X \to Y_1 \times_{\mathbb{P}^1} Y_2$, using the complete classification of rational quasi-elliptic surfaces with section up to isomorphism in $p = 2, 3$:

Theorem 1.3 ([Ito92],[Ito94]). *A rational quasi-elliptic surface with section is given by one of the following:*

Type of degenerate fibers	Weierstrass form
$p = 3$	
(a) II^*	$y^2 = x^3 + t$
(b) $\mathrm{IV}, \mathrm{IV}^*$	$y^2 = x^3 + t^2$
(c) *Four* IV*'s*	$y^2 = x^3 + t^4 + t^2$
$p = 2$	
(a) II^*	$y^2 = x^3 + t^5$
(b) I_4^*	$y^2 = x^3 + t^2x + t^5$
(c) $\mathrm{III}, \mathrm{III}^*$	$y^2 = x^3 + t^3x$
(d) *Two* I_0^**'s*	$y^2 = x^3 + at^2x + t^3,\ a \in k$
(e) I_2^*, *two* III*'s*	$y^2 = x^3 + (t^3 + t)x$
(f) I_0^*, *four* III*'s*	$y^2 = x^3 + (t^3 + at^2 + t)x,\ a \in k^*$
(g) *Eight* III*'s*	$y^2 = x^3 + (t^3 + at^2 + bt)x + t^3,\ a \in k,\ b \in k^*$

where II^*, IV *and* IV^* *stand for the types of singular fibers in the sense of Kodaira.*

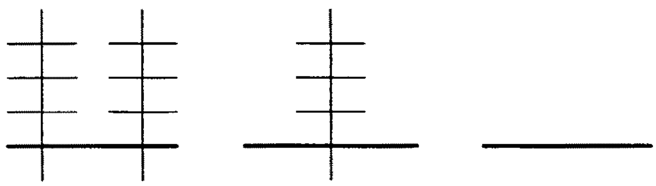
One of the main differences between Schoen's Calabi-Yau 3-folds and ours is the complexity of the singularities of $Y_1 \times_{\mathbb{P}^1} Y_2$. Since $\mathrm{Sing}\,(Y_1 \times_{\mathbb{P}^1} Y_2)$ comes from the non-smooth parts of φ_1 and φ_2, it consists of irreducible curves isomorphic to $\mathbb{P}^1$. The calculation of a resolution is very tedeous, especially in characteristic 2.

2. Construction in characteristic 3

In this report, we treat mainly the case when the characteristic of the base field is 3. In order to obtain smooth Calabi-Yau 3-folds, we need the following conditions:

(1) We do not use quasi-elliptic surfaces of type (a) in Theorem 1.3 as Y_1 or Y_2.
(2) The singular fiber of type IV^* on a quasi-elliptic surface, does not meet any special fiber on the other one.

Thus the choices of two surfaces as Y_1 and Y_2 are three: (b)-(b), (b)-(c), or (c)-(c). The configuration of the singularities of $Y_1 \times_{\mathbb{P}^1} Y_2$ is as in Figure 1. Note that the thick lines, which will be denoted by Γ, are derived from the moving cusps of two quasi-elliptic surfaces.



Case (bb) Case (bc) Case (cc)

FIGURE 1

We have eight sub-cases under the condition (2):

(bb-1) the singular fiber of type IV meets the singular fiber of type IV,
(bb-2) the singular fiber of type IV does not meet the singular fiber of type IV,
(bc-1) the singular fiber of type IV meets a singular fiber of type IV,
(bc-2) the singular fiber of type IV does not meet any singular fiber of type IV,
(cc-1) four singular fibers of type IV meet singular fibers of type IV,
(cc-2) two singular fibers of type IV meet singular fiber of type IV,
(cc-3) one singular fiber of type IV meets singular fiber of type IV,
(cc-4) no singular fiber of type IV meets singular fibers of type IV.

For local calculation, we need the following proposition:

Proposition 2.1. *Let $\varphi : Y \to C$ be a relatively minimal quasi-elliptic surface in characteristic 3. We take a point P on Y and any local coordinate t on C at $\varphi(P)$.*

(1) [BM76] *Suppose that P lies on the moving cusp Σ. If the fiber over $t = 0$ is nonspecial, then in suitable formal coordinates x, y on Y at P, we have $t = y^2 + x^3$.*
(2) *Suppose that P lies on the moving cusp Σ. If the fiber over $t = 0$ is of type* IV*, then in suitable formal coordinates x, y on Y at P, we have $t = xy^2 - x^3$.*
(3) *Suppose that the fiber over $t = 0$ is of type* IV**. If P is an intersection point of the component of multiplicity three and a component of multiplicity two (resp. the moving cusp Σ), then there exist formal coordinates x, y such that $t = x^3y^2$ (resp. $t = x^3(1 + y^2)$). If P is on the component of multiplicity three but outside the four points described above, then $t = (1 + y)x^3$.*

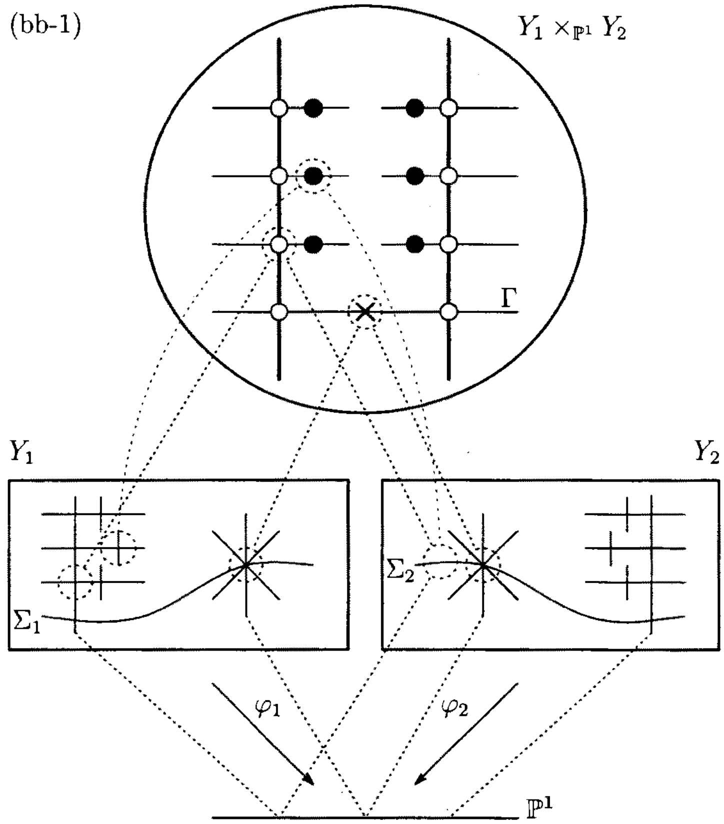
For example, take the case (bb-1). The singularities come from the cusp of a general fiber and components of the singular fiber of type IV* whose multiplicities are greater than one. At a general point on Γ which projects to both cusps of two general fibers of φ_1 and φ_2, we have the equation $x^3 + y^2 + z^3 + w^2 = 0$, by Proposition 2.1 (1). Since we are in characteristic 3, we obtain a local equation

$$x^3 + y^2 + z^2 = 0.$$

At a general point of the curve (expressed as the thick line in Figure 1) which comes from the moving cusp of Y_1 and the component of multiplicity three of the singular fiber in Y_2, we have the equation $x^3 + y^2 + z^3(1 + w^2) = 0$ by Proposition 2.1 (1) and (3). Hence we obtain a local equation

$$x^3 + y^2 + z^3w = 0.$$

We can also determine other local equations of the singularities in a similar way. Thus we have the structure diagram of the singularities such as Figure 2.



: $x^3 + y^2 + z^2 = 0$
: $x^3 + y^2 + z^3 w = 0$
: $x^3 + y^2 + z^3 w^2 = 0$
: $x^3 + y^2 + z^2 w = 0$
: $x^3 + xy^2 + y^2 z + zw^2 = 0$

FIGURE 2

Now we can resolve these singularities. Local calculation shows the resolution is crepant. In (1) and (4), blowing up with the center of the reduced singular locus $\{x = y = z = 0\}$ gives a resolution. In (2), blow up the reduced singular locus $\{x = y = z = 0\}$. There appears a one dimensional singular locus which is locally a trivial deformation of a rational double point of type A_1. Blowing up this singular locus gives a resolution. In (3), one can reduce to the case of type (2) after a blow-up along $\{x = y = w = 0\}$. In (5), blow up $\{x = y = w = 0\}$, there remain six ordinary double points. The inverse image of the origin is $\mathbb{P}^2$ and blowing up this $\mathbb{P}^2$ gives a small resolution.

Since $Y_1 \times_{\mathbb{P}^1} Y_2$ is a divisor of a nonsingular fourfold $Y_1 \times_k Y_2$, all the singularities of $Y_1 \times_{\mathbb{P}^1} Y_2$ are hypersurface singularities. In characteristic zero, any isolated singularity in codimension two is generically a trivial deformation of a rational double point if it has a crepant resolution ([Rei80, Corollary 1.14]). But in positive characteristic, this is not always the case.

3. Supersingularity and topological invariants

Calabi-Yau 3-folds we constructed in the previous section have some 'peculiar' properties mentioned in the Introduction.

Supersingularity. Since the base change of a quasi-elliptic surface $\varphi : Y \to \mathbb{P}^1$ by the Frobenius morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is a non-normal rational surface, X has the fibration $f : X \to \mathbb{P}^1$ induced from $\varphi_1 \times_{\mathbb{P}^1} \varphi_2$. Then the base change $\tilde{X} := X \times_{\mathbb{P}^1} \mathbb{P}^1$ of f by the Frobenius morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is a rational threefold. Therefore X is purely inseparably unirational, hence supersingular.

Non-liftability. Let Z be the normalization of the base change of $Y_1 \times_{\mathbb{P}^1} Y_2$ by Frobenius morphism $F : \mathbb{P}^1 \to \mathbb{P}^1$.

$$\begin{tikzcd}
 & & X \arrow[d, "\pi"] & \\
Z \arrow[rr] \arrow[ddd] & & Y_1 \times_{\mathbb{P}^1} Y_2 \arrow[dl] \arrow[dr] & \\
 & Y_1 \arrow[dr, "\varphi_1"] & & Y_2 \arrow[dl, "\varphi_2"'] \\
 & & & \\
\mathbb{P}^1 \arrow[rr, "F"] & & \mathbb{P}^1 &
\end{tikzcd}$$

We can calculate the topological Euler-Poincaré characteristic $e(Z)$ of Z, which is equal to $e(Y_1 \times_{\mathbb{P}^1} Y_2)$. By argument of divisors on X, we obtain also the Picard number $\rho(X)$.

Proposition 3.1. *The Calabi-Yau threefolds obtained in the previous section have the following invariants.*

	(bb-1)	(bb-2)	(bc-1)	(bc-2)	(cc-1)	(cc-2)	(cc-3)	(cc-4)
$e(X)$	72	60	60	48	84	60	48	36
$\rho(X)$	35	30	30	25	41	30	25	20

As a corollary of the proposition above, we have $b_3(X) = 0$ for the cases (bb-1) and (cc-1), which implies non-liftability of X to characteristic zero.

4. SUPERSINGULAR K3 FIBRATION

Since each Y_i is a rational quasi-elliptic surface, it has a $\mathbb{P}^1$-fibration $\tau_i : Y_i \longrightarrow \mathbb{P}^1$. We examine the fibration structures induced by τ_i of Calabi-Yau 3-folds we constructed in the previous sections.

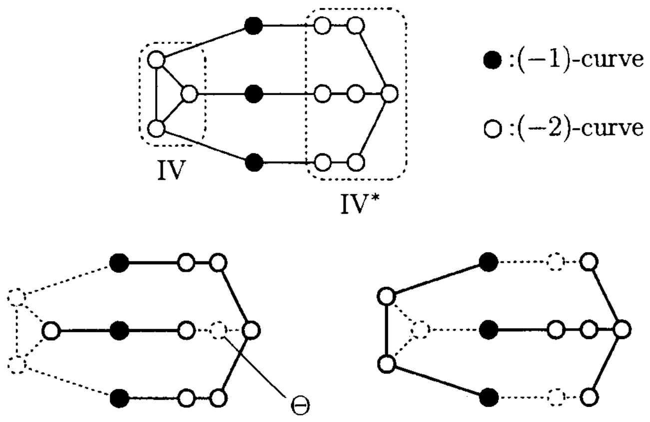
$$\begin{tikzcd}
 & & X \arrow[d, "\pi"] \arrow[ddddll, bend right, "g"'] & & \\
 & & Y_1 \times_{\mathbb{P}^1} Y_2 \arrow[dl] \arrow[dr] & & \\
 & Y_1 \arrow[d, "\psi_1"] \arrow[ddl, "\tau_1"'] \arrow[ddr, "\varphi_1"] & & Y_2 \arrow[ddl, "\varphi_2"'] \arrow[ddr, "\tau_2"] & \\
 & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl, "\mathrm{pr.}"] \arrow[dr, "\mathrm{pr.}"'] & & & \\
\mathbb{P}^1 & & \mathbb{P}^1 & & \mathbb{P}^1
\end{tikzcd}$$

FIGURE 3

We denote by $g : X \longrightarrow \mathbb{P}^1$ the composition morphism $X \xrightarrow{\pi} Y_1 \times_{\mathbb{P}^1} Y_2 \overset{\mathrm{proj}_1}{\longrightarrow} Y_1 \xrightarrow{\tau_1} \mathbb{P}^1$ (cf. Figure 3). Let F_{φ_1} and F_{τ_1} be general fibers of $\varphi_1 : Y_1 \longrightarrow \mathbb{P}^1$ and $\tau_1 : Y_1 \longrightarrow \mathbb{P}^1$, respectively. By the canonical bundle formula for Y_1, we observe $F_{\varphi_1}.F_{\tau_1} = 2$. This means that a general fiber of the composition $Y_1 \times_{\mathbb{P}^1} Y_2 \overset{\mathrm{proj}_1}{\longrightarrow} Y_1 \xrightarrow{\tau_1} \mathbb{P}^1$ is obtained as the pull-back of Y_2 by a double cover $\varphi_1|_{F_{\tau_1}} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$, which is ramified at two points by Hurwitz formula. Thus taking a double cover $\psi_1 := (\varphi_1, \tau_1) : Y_1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$, we investigate the ramification divisor R_1 of ψ_1. Note that the configurations of degenerate fibers and sections on rational quasi-elliptic surfaces of type (b) and (c) are given in Theorem 1.3.

Here we illustrate the configuration of all the (-1)-curves and (-2)-curves on Y_1 as the dual graph in the case (b). The structure of $g : X \longrightarrow \mathbb{P}^1$ depends on which curves are contained in the fibers of τ_1. Finding such subgraphs in the dual graph, we can determine all the $\mathbb{P}^1$-fibrations on Y_1 which are essentially classified into two classes (cf. Figure 4).

For the left case in Figure 4, we can check that Θ is a component of R_1. Moreover, since $\Sigma.F_{\tau_1} = 1$ and $F_{\varphi_1}.F_{\tau} = 2$ where Σ is the moving cusp of Y_1, the double cover ψ_1 is also ramified along Σ. We can see that a general fiber of $Y_1 \times_{\mathbb{P}^1} Y_2 \longrightarrow \mathbb{P}^1$ has two RDPs of type A_2 at the points that ramification divisor R_1 penetrates. Thus taking a crepant resolution $\pi : X \longrightarrow Y_1 \times_{\mathbb{P}^1} Y_2$, we obtain a smooth K3 surface as a general fiber of $g : X \longrightarrow \mathbb{P}^1$, which is supersingular since it has a quasi-elliptic fibration.



Subgraphs emphasized stand for components in the fibers of τ_1.

FIGURE 4

On the other hand, for the right case in Figure 4, no fibers of $\varphi_1 : Y_1 \to \mathbb{P}^1$ has a component of R_1. In this case, a general fiber of $g : X \to \mathbb{P}^1$ has an RDP of type A_2.

5. CHARACTERISTIC 2 CASE

We can also construct Calabi-Yau 3-folds with 'peculiar' properties in characteristic 2. Methods are almost all the same as in characteristic 3, but local calculation is much more complicated. Here we only review the result.

The pairs of quasi-elliptic surfaces we consider as Y_1 and Y_2 to construct smooth Calabi-Yau 3-folds are as follows:
(bb): (b) and (b), I_4^* does not meet I_4^*,
(bc): (b) and (c), I_4^* meets III,
(bd): (b) and (d), I_4^* does not meet I_0^*,
(be): (b) and (e), I_4^* meets III,
(dd): (d) and (d), I_0^* does not meet I_0^*,
(de): (d) and (e), both I_0^* meet III's.

For these six cases, we can determine the singularities on $Y_1 \times_{\mathbb{P}^1} Y_2$ and resolve them to obtain smooth Calabi-Yau 3-folds. By calculating the topological invariants, one can see that Calabi-Yau's of types (bb) and (bc) are not liftable to characteristic zero.

Figure 5 gives the configuration of all the (−1)-curves and (−2)-curves on Y_1 and the in the cases (b), (c), (d) and (e). The subgraphs emphasized stand for curves contained in the fibers of τ_1. Let $Y_1 \to S \to \mathbb{P}^1 \times \mathbb{P}^1$ be the Stein factorization of ψ_1. Since we are in characteristic 2, the double cover $S \to \mathbb{P}^1 \times \mathbb{P}^1$ might be inseparable.

Our results are as follows:

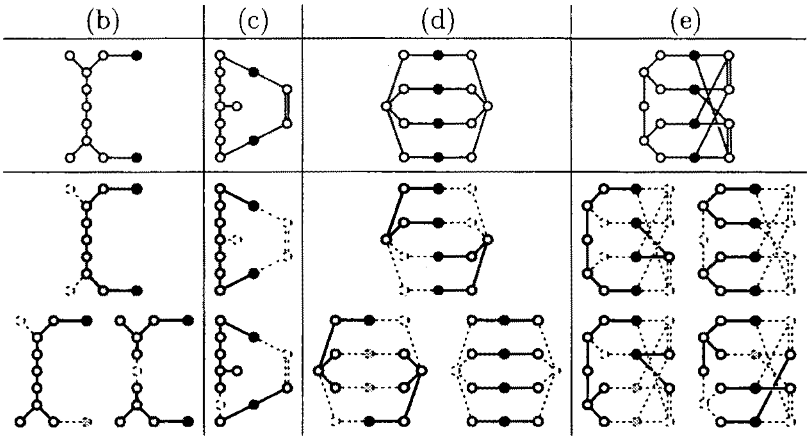


FIGURE 5

- The case $\rho(S) = 2$.
  In this case, $S \to \mathbb{P}^1 \times \mathbb{P}^1$ is inseparable, and a general fiber of $g : X \to \mathbb{P}^1$ is a non-normal rational surface. These correspond to the cases of the lowest columns in Figure 5.
- The case $\rho(S) > 2$ and $\tau_1 = \Phi_{|\Sigma|}$ (which occurs only when Y_1 is of type (c) or (e)).
  In this case, $S \to \mathbb{P}^1 \times \mathbb{P}^1$ is separable, and the ramification locus corresponds to the irreducible component of III^* or I_2^* which intersects the moving cusp of Y_1. In this case, a general fiber of $g : X \to \mathbb{P}^1$ is a smooth supersingular K3 surface.
- The case $\rho(S) > 2$ and $\tau_1 \neq \Phi_{|\Sigma|}$.
  In this case, $S \to \mathbb{P}^1 \times \mathbb{P}^1$ is separable, and the ramification locus corresponds to the moving cusp of Y_1. In this case, a general fiber of $g : X \to \mathbb{P}^1$ is
  - a smooth supersingular K3 surface if Y_1 is of type (b) or (d), and
  - a supersingular K3 surface with three RDPs of type A_1 if Y_1 is of type (e).

For the details, see [HIS].

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